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# Eleven-dimensional Lorentz symmetry from SUSY quantum mechanics

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Abstract: The supermembrane in light-cone gauge gives rise to a supersymmetric quantum mechanics system with $SU(N)$ gauge symmetry when the group of area preserving diffeomorphisms is suitably regulated. de Wit, Marquard and Nicolai showed how eleven-dimensional Lorentz generators can be constructed from these degrees of freedom at the classical level. In this paper, these considerations are extended to the quantum level and it is shown the algebra closes to leading nontrivial order at large $N$. A proposal is made for extending these results to Matrix theory by realizing longitudinal boosts as large $N$ renormalization group transformations.

Keywords: M-Theory, M(atrix) Theories, Space-Time Symmetries.

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## 1. Introduction

Some time ago, de Wit, Hoppe and Nicolai [1] showed that the supermembrane in light-cone gauge reduces to maximally supersymmetric quantum mechanics with $SU(N)$ gauge symmetry in the large $N$ limit. The $SU(N)$ gauge symmetry arises as a regulated version of the area preserving diffeomorphism symmetry of the supermembrane. More recently [2] Banks, Fischler, Shenker and Susskind proposed to use this supersymmetric quantum mechanics as a complete description of eleven-dimensional M-theory. This description has come to be known as Matrix theory. The key distinction between this new point of view and the previous work is the identification of $N$ with the number of quanta of momentum along a compact direction. The relationship between supersymmetric quantum mechanics and M-theory was clarified in [3]. There it was argued that if one assumes eleven-dimensional Lorentz invariance, and the duality between Type IIA string theory and M-theory on a spacelike circle, M-theory on a light-like circle reduces to the supersymmetric quantum mechanics system. The limiting procedure needed for this argument to work has been further studied in [4].

In this paper we will take a different point of view and study how eleven-dimensional Lorentz invariance can be recovered, starting with the supersymmetric quantum mechanics. A number of scattering amplitude calculations already provide evidence for the Lorentz invariance of Matrix theory at large $N$. See the review [5] for a discussion of some of these results and further references. A notable recent example was the computation using Matrix theory of the three graviton scattering amplitude, in agreement with the prediction of supergravity [6].

Ideally one would like a direct construction of the Lorentz generators using the quantum mechanics degrees of freedom. In the original dWHN [1] interpretation of the supersymmetric quantum mechanics, these Lorentz generators where constructed in [7] and shown to close at the classical level, up to $1/N^2$ corrections [7, 8]. On the other hand, for finite $N$ Matrix theory the construction of such generators seems difficult since longitudinal boosts change the value of $N$. Suitable definitions should exist for large $N$, as will be discussed in the following.

In this paper we extend the analysis of the dWHN model to the quantum theory by properly dealing with ordering ambiguities, and find the Lorentz algebra closes to leading nontrivial order at large $N$. We propose a way to extend these results to Matrix theory, by realizing longitudinal boosts as large $N$ renormalization group transformations.



## 2. Supermembranes and SUSY quantum mechanics

In this section we review the relationship between the light-cone formulation of the supermembrane and supersymmetric quantum mechanics [1]. For the most part we will follow the notation of [7]. After gauge fixing, the Lagrangian takes the form

$$\frac{1}{\sqrt{w}}\mathcal{L} = \frac{1}{2}(D_0\vec{X})^2 + \frac{i}{2}\theta D_0\theta - \frac{1}{4}\left(\left\{X^a, X^b\right\}\right)^2 + \frac{i}{2}\theta\gamma_a\left\{X^a,\theta\right\}, \tag{2.1}$$

where $X^a(t,\sigma^r)$, $\theta_\alpha(t,\sigma^r)$ ($a = 1,\ldots,9$, $\alpha = 1,\ldots,16$, $r = 1,2$) are the transverse worldvolume degrees of freedom dependent on the worldvolume coordinates $t$ and $\sigma^r$. The indices $a$ and $\alpha$ are respectively the vector and the spinor degrees of freedom of $SO(9)$. The conventions for gamma matrices are described in Appendix B. $w_{ij}$ is the $2\times 2$ spatial metric tensor on the worldvolume and $w$ is its determinant. The bracket is defined as, $\{A,B\} \equiv \frac{\epsilon^{rs}}{\sqrt{w(\sigma)}}\partial_r A(\sigma)\partial_s B(\sigma)$. The covariant derivative, $D_0X^a = \partial_0 X^a - \{\omega, X^a\}$, $D_0\theta = \partial_0\theta - \{\omega,\theta\}$, defines the gauge transformation corresponding to an area preserving diffeomorphism (APD),

$$\delta X^a = \{\xi, X^a\}, \quad \delta\theta = \{\xi,\theta\}, \quad \delta\omega = \partial_0\xi + \{\xi,\omega\}. \tag{2.2}$$

The canonical Hamiltonian is

$$\begin{aligned} H &= -\int d^2\sigma P^-(\sigma) \\ &= \frac{1}{P_0^+}\int d^2\sigma\sqrt{w(\sigma)}\left(\frac{1}{2}w^{-1}\vec{P}^2 + \frac{1}{4}\left(\left\{X^a,X^b\right\}\right)^2 - \frac{i}{2}\theta\gamma_a\left\{X^a,\theta\right\}\right). \end{aligned} \tag{2.3}$$

$\vec{P}$ denotes the canonical momentum conjugate to $\vec{X}$. The non-vanishing Dirac brackets are

$$\begin{aligned}
\left(X^a(\sigma), P^b(\rho)\right)_{DB} &= \delta^{ab}\delta^{(2)}(\sigma,\rho), \\
(\theta_\alpha(\sigma), \theta_\beta(\rho))_{DB} &= -\frac{i}{\sqrt{w(\sigma)}}\delta_{\alpha\beta}\delta^{(2)}(\sigma,\rho)\,.
\end{aligned} \tag{2.4}$$

The Gauss law constraints associated with the APD are

$$\varphi(\sigma) \equiv -\left\{\frac{P^a(\sigma)}{w(\sigma)}, X^a(\sigma)\right\} - \frac{i}{2}\{\theta(\sigma),\theta(\sigma)\} \approx 0\,. \tag{2.5}$$

Here we take the membrane to have spherical topology. For higher topology there are additional APD's generated by the harmonic vectors, which give rise to additional Gauss law constraints.

The light cone directions are defined as $X^\pm = \frac{1}{\sqrt{2}}(X^{10} \pm X^0)$ where $X^+$ is related to the worldvolume time $t$ by $X^+(\tau) = X^+(0) + t$, and

$$\partial_r X^-(\sigma) = -\frac{1}{P_0^+}\left(\frac{1}{\sqrt{w(\sigma)}}\vec{P}(\sigma)\cdot\partial_r\vec{X}(\sigma) + \frac{i}{2}\theta(\sigma)\partial_r\theta(\sigma)\right). \tag{2.6}$$

The integrability conditions of this differential equation coincide with the Gauss law constraints. When integrated, it gives

$$X^-(\sigma) = q^- - \frac{1}{P_0^+}\int d^2\rho G^r(\sigma,\rho)\left(\vec{P}(\rho)\cdot\partial_r\vec{X}(\rho) + \frac{i}{2}\sqrt{w(\rho)}\theta(\rho)\partial_r\theta(\rho)\right), \tag{2.7}$$

where the integration constant satisfies $(q^-, P_0^+)_{DB} = 1$ and $G^r(\sigma,\rho)$ is the Green function defined by $D_r^\rho G^r(\sigma,\rho) = -(w(\sigma))^{-1/2}\delta^{(2)}(\sigma,\rho) + 1$.

This system has supersymmetry generated by

$$\begin{aligned}
Q^+ &= \frac{1}{\sqrt{P_0^+}}\int d^2\sigma\left(P^a\gamma_a + \frac{\sqrt{w}}{2}\left\{X^a, X^b\right\}\gamma_{ab}\right)\theta, \\
Q^- &= \sqrt{P_0^+}\int d^2\sigma\sqrt{w(\sigma)}\theta\,.
\end{aligned} \tag{2.8}$$

The Lorentz generators are defined by

$$\begin{aligned}
M^{ab} &= \int d^2\sigma\left(-P^aX^b + P^bX^a - \frac{i}{4}\theta\gamma^{ab}\theta\right), \\
M^{+-} &= \int d^2\sigma\left(-P^+X^- + P^-X^+\right), \\
M^{+a} &= \int d^2\sigma(-P^+X^a + P^aX^+), \\
M^{-a} &= \int d^2\sigma\left(P^aX^- - P^-X^a - \frac{i}{4P_0^+}\theta\gamma^{ab}\theta P_b - \frac{i\sqrt{w}}{8P_0^+}\left\{X_b, X_c\right\}\theta\gamma^{abc}\theta\right).
\end{aligned} \tag{2.9}$$



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To regulate this theory we follow [1], and expand all the worldvolume fields in terms of spherical harmonics, with a mode cutoff dependent on $N$. The eigenfunctions satisfy

$$\Delta Y_0 = 0, \quad \Delta Y_A = -\omega_A Y_A\,, \tag{2.10}$$

with $\omega_A > 0$. Indices $A, B, C$ are positive integers, $I, J, K$ denote non-negative integers. The $Y_I$ are orthonormal

$$\int d^2\sigma \sqrt{w(\sigma)} Y^I(\sigma) Y_J(\sigma) = \delta^I{}_J \qquad Y^I \equiv Y_I^* = \eta^{IJ} Y_J\,. \tag{2.11}$$

The completeness relations take the form

$$\sum_A Y^A(\sigma) Y_A(\rho) = \frac{1}{\sqrt{w(\sigma)}} \delta^{(2)}(\sigma, \rho) - 1\,, \tag{2.12}$$

and

$$\begin{aligned} \sum_A \frac{1}{\omega_A} [D^r Y_A(\sigma) D^s Y^A(\rho) + \frac{\epsilon^{rt}}{\sqrt{w(\sigma)}} \partial_t Y_A(\sigma) \frac{\epsilon^{su}}{\sqrt{w(\rho)}} \partial_u Y^A(\rho)] = \\ = \frac{w^{rs}(\sigma)}{\sqrt{w(\sigma)}} \delta^{(2)}(\sigma, \rho)\,. \end{aligned} \tag{2.13}$$

The Green function is rewritten,

$$G^r(\sigma, \rho) = \sum_A \frac{1}{\omega_A} Y^A(\sigma) \partial^r Y_A(\rho)\,. \tag{2.14}$$

Three-index tensors that will be used in the following are:

$$\begin{aligned} f_{ABC} &= \int d^2\sigma \sqrt{w(\sigma)} Y_A(\sigma) \left\{ Y_B(\sigma), Y_A(\sigma) \right\}, \\ d_{ABC} &= \int d^2\sigma \sqrt{w(\sigma)} Y_A(\sigma) Y_B(\sigma) Y_C(\sigma)\,, \\ c_{ABC} &= -2 \int d^2\sigma \sqrt{w(\sigma)} \frac{w^{rs}}{\omega_A} \partial_r Y_A Y_B \partial_s Y_C\,. \end{aligned} \tag{2.15}$$

These tensors satisfy a number of nontrivial identities which are described in appendix A.

The approximation of the group of area preserving diffeomorphisms on $S^2$ by the group $SU(N)$ is considered in detail in appendix B of [7] (see also references therein). Here we will only need to know that $A = 1, \cdots, N^2 - 1$ and that as $N \to \infty$ $f_{ABC}$ is well approximated by the structure constants of $SU(N)$, $d_{ABC}$ is proportional to the three index symmetric tensor, and $c_{ABC}$ is a linear combination of the invariant tensors $f_{ABC}$ and $d_{ABC}$. It should be pointed out that $c_{ABC}$ is not invariant under APD's [7], since it explicitly depends on the worldvolume metric. However once the group of APD's is regulated via $SU(N)$ it is possible to modify the definition of $c_{ABC}$ so that it is invariant under $SU(N)$. Even with this modification the components of $c_{ABC}$

are inherently ambiguous up to terms of order $1/N^2$. For concreteness we follow the definitions of [7], appendix B. With this approximation all the identities of appendix A are satisfied up to $1/N^2$ corrections. These corrections are harmless at the order to be considered in this paper.

Now the membrane formulae above may be rewritten in terms of $SU(N)$ variables and the zero modes. The Gauss law constraints are re-expressed as

$$\varphi_A = f_{ABC}\left(\vec{X}^B \cdot \vec{P}^C - \frac{i}{2}\theta^B\theta^C\right). \tag{2.16}$$

Expressing the Dirac brackets as commutators we have:

$$\begin{aligned} [X_a^A, P_{bB}] &= i\delta_{ab}\delta_B^A & \{\theta_\alpha^A, \theta_{\beta B}\} &= \delta_{\alpha\beta}\delta_B^A \\ [q^-, P_0^+] &= i & [X_{a0}, P_{b0}] &= i\delta_{ab} \\ \{\theta_{\alpha 0}, \theta_{\beta 0}\} &= \delta_{\alpha\beta}\,. \end{aligned} \tag{2.17}$$

The Hamiltonian takes the form

$$\begin{aligned} H &= \frac{\vec{P}_0^2}{2P_0^+} + \frac{\mathcal{M}^2}{2P_0^+} \\ \mathcal{M}^2 &= \vec{P}_A^2 + \frac{1}{2}(f_{ABC}X_a^BX_b^C)^2 - if_{ABC}\theta^A\gamma^aX_a^B\theta^C\,. \end{aligned} \tag{2.18}$$

The $X^-$ coordinate is rewritten

$$X_A^- = -\frac{1}{P_0^+}(\vec{P}_0\cdot\vec{X}_A + \frac{i}{2}\theta_0\theta_A) + \frac{1}{2P_0^+}c_{ABC}(\vec{P}^B\cdot\vec{X}^C + \frac{i}{2}\theta^B\theta^C)\,. \tag{2.19}$$

The Lorentz generators at $t = 0$ are [7]:

$$\begin{aligned} M^{ab} &= -P_0^aX_0^b + P_0^bX_0^a - \frac{i}{4}\theta_0\gamma^{ab}\theta_0 - P_A^aX^{bA} + P_A^bX^{aA} - \frac{i}{4}\theta_A\gamma^{ab}\theta^A \\ M^{+a} &= -P_0^+X_0^a \\ M^{+-} &= -P_0^+q^- + A \\ M^{-a} &= (M^{-a})^{(0)} + \frac{1}{P_0^+}(P_{0b}\tilde{M}^{ab} - \frac{i}{2}\theta_0\gamma^a\tilde{Q}^+) + \frac{1}{P_0^+}\tilde{M}^{-a}\,, \end{aligned} \tag{2.20}$$

where we have defined

$$\begin{aligned} (M^{-a})^{(0)} &= q^-P_0^a + X_0^aH - \frac{i}{4P_0^+}\theta_0\gamma^{ab}\theta_0P_{0b} + B\frac{P_0^a}{P_0^+}\,, \\ \tilde{Q}^+ &= (P_A^a\gamma_a + \frac{1}{2}f_{ABC}X_a^BX_b^C\gamma^{ab})\theta^A\,, \\ \tilde{M}^{ab} &= -P_A^aX^{bA} + P_A^bX^{aA} - \frac{i}{4}\theta_A\gamma^{ab}\theta^A\,, \\ \tilde{M}^{-a} &= \frac{1}{2}d^{ABC}X_A^a(P_B\cdot P_C + \frac{1}{2}(f_B{}^{DE}X_D^bX_E^c)(f_C{}^{FG}X_F^bX_G^c) - if_C{}^{DE}X_D^b\theta_B\gamma_b\theta_E) - \\ &\quad -\frac{i}{4}d^{ABC}P_{Ab}\theta_B\gamma^{ab}\theta_C + \frac{1}{2}c^{ABC}P_A^a(P_B\cdot X_C + \frac{1}{2}i\theta_B\theta_C) - \\ &\quad -\frac{i}{8}f^{ABC}d_A{}^{DE}X_{Bb}X_{Cc}\theta_D\gamma^{abc}\theta_E\,. \end{aligned} \tag{2.21}$$

We adopt an ordering prescription that preserves the $SO(9)$ rotational symmetry and take all products of (fermionic) bosonic operators to be (a)symmetrically ordered. In (2.18), (2.20), (2.21) the possible ordering terms have been included. The terms with coefficients $A$ and $B$ are the only terms allowed by the symmetries.

$A$ is fixed by demanding that $[M^{+a}, M^{-a}] = i\eta^{aa}M^{+-}$ is satisfied. This fixes $A = 0$. To fix $B$ we consider $[M^{-a}, X_0^a] \sim q^- + P_0^a/P_+$. To remove the inhomogeneous term we need to set $B = i$. This follows from the fact that $X_0^a H + iP_0^a/P_+$ with symmetric ordering equals $X_0^a H$ without symmetric ordering.

## 3. Lorentz algebra in quantum theory

The Lorentz algebra has been considered at the classical level in [7, 8], where it was found the algebra closes up to $1/N^2$ corrections. Typically, ordering terms appear with extra factors of $N^2$ arising from the trace over group generators. We will show the Lorentz algebra closes at the quantum level up to order 1 ordering terms. To go beyond this calculation would require a computation at the classical level to the next nontrivial order. It is straightforward to see that the symmetries allow nontrivial ordering terms only in the commutators $[M^{-a}, M^{-b}]$ and $[M^{-a}, H]$. In the following we compute the ordering terms for these commutators. The terms that have already been shown to vanish at the classical level [7, 8] will not be discussed further.

One might expect that these ordering terms will only vanish in the critical dimension of the supermembrane [9], as in the analogous light-cone string theory calculation. In that case, the vanishing of the normal ordered commutator $[M^{-a}, M^{-b}]$ led to a critical dimension of ten. Here we will find the gauge symmetry is much more restrictive than in the string case, and the ordering terms will vanish identically at leading nontrivial order at large $N$ for any dimension in which the classical algebra closes (although our explicit calculations are for the case of eleven dimensions only). Certain spinor identities (see appendix B) are needed to prove the closure of the algebra at the classical level. These restrict the possible allowed dimensions to $4, 5, 7$ or $11$. Of course the BFSS interpretation of the supersymmetric quantum mechanics can only hold in eleven dimensions because only in that case do we get a normalizable ground state for the $N = 2$ system corresponding to a graviton with two units of longitudinal momentum [10].

### 3.1. $[M^{-a}, M^{-b}]$

The nontrivial ordering contributions come from the following terms, which are written without (a)symmetric ordering

$$[M^{-a}, M^{-b}] = \frac{1}{(P_0^+)^2}(A_1 + A_2 + A_3 + A_4)\,, \tag{3.1}$$

with

$$A_1 = -\frac{i}{2}(\mathcal{M}^2\tilde{M}^{ab} + \tilde{M}^{ab}\mathcal{M}^2)$$

$$
\begin{aligned}
A_2 &= \frac{1}{4}\tilde{Q}^+\gamma^{ab}\tilde{Q}^+ \\
A_3 &= [\tilde{M}^{-a}, \tilde{M}^{-b}] \\
A_4 &= \frac{1}{2}\theta_0\gamma^a[\tilde{Q}^+, \tilde{M}^{-b}] - (a \leftrightarrow b)\,.
\end{aligned} \tag{3.2}
$$

It is easy to see ordering terms of order $P^2$ vanish by symmetry considerations alone. Let us ignore for the moment the fermionic terms. The other bosonic terms arise from ordering expressions of order $PX^5$. We choose to order these terms so the $P$ factors are on the left. When we move the $P$ factor to the left in (3.2), $X^4$ terms are generated. The reordered $PX^5$ terms combine according to the classical analysis of [8]. We must show the extra terms generated by this reordering cancel.

The terms appearing in $A_1$ are proportional to

$$
f_{ABC}f^{A}_{\ DE}X^{Bb}X^C_d X^{Da}X^{Ed} - (a \leftrightarrow b)\,, \tag{3.3}
$$

which vanishes since the first term is symmetric under interchange of $a$ and $b$.

To extract the ordering terms that arise in $A_3$ it is helpful to use the analysis of [8] as much as possible. Following through the steps in that calculation, the computation of the ordering term amounts to ordering eq. (82) of [8]

$$
\begin{aligned}
&\frac{1}{2}(f_{ABC}X^B_d X^C_e)^2(X^a_D P^{bD} - X^b_D P^{aD}) + \\
&+\frac{1}{2}d^{AIC}f_I^{\ DE}(X^a_A X^b_E - X^b_A X^a_E)\vec{X}_D\cdot\vec{X}_F c_{HC}^{\ \ F} f^{HBG}\vec{X}_G\cdot\vec{P}_B + \\
&+f^{DAE}(X^a_A X^b_E - X^b_A X^a_E)\vec{X}_D\cdot\vec{X}_F f^{FBG}\vec{X}_G\cdot\vec{P}_B\,.
\end{aligned} \tag{3.4}
$$

Ordering the first term gives no extra terms by the same calculation as for $A_1$. Likewise the same tensor structures appear when ordering the third term, so that also vanishes. The second term gives rise to the ordering terms

$$
\begin{aligned}
(d^{AIC}c^H_{\ CF}f_{IDE}f_{HJ}^{\ \ F} + d^{FIC}c_{HCJ}f_{IDE}f^{HA}_{\ \ F} - d^{AIC}c_{HCJ}f_{IDF}f_E^{\ HF} - \\
-d^{AIC}c_{HCJ}f_{IFE}f_D^{\ HF})X^a_A X^{Eb}X^D\cdot X^J - (A \leftrightarrow E)\,.
\end{aligned} \tag{3.5}
$$

Applying the Jacobi identity A.1$a$ and the identity A.1$e$ we can reduce (3.5) to

$$
(d^{AIC}c^H_{\ CF}f_{IDE}f_{HJ}^{\ \ F} - d^{FIA}c^H_{\ CJ}f_{EDF}f_{HI}^{\ \ C})X^a_A X^{Eb}X^D\cdot X^J - (A \leftrightarrow E)\,. \tag{3.6}
$$

Using the symmetries and A.1$b$ we find these terms cancel. This proves no extra ordering terms arise and the classical computation of [8], goes through, with the understanding that $P$'s are ordered to the left. They find these terms in the commutator combine to give (see eq. (88) [8])

$$
\begin{aligned}
&-\frac{1}{2}d^{AIC}f_I^{\ DE}c_{HC}^{\ \ F}f^{HBG}\vec{P}_B\cdot\vec{X}_G(X^a_A X^b_E - X^b_A X^a_E)\vec{X}_D\cdot\vec{X}_F - \\
&-f^{DAE}f^{FBG}\vec{P}_B\cdot\vec{X}_G(X^a_A X^b_E - X^b_A X^a_E)\vec{X}_D\cdot\vec{X}_F
\end{aligned} \tag{3.7}
$$

which are proportional to the purely bosonic part of the constraints. The next step is to check that no extra terms are generated when this expression is symmetrically ordered. The two terms in (3.7) are proportional to the second and third terms in (3.4) and we have already shown the ordering terms in these expressions vanish. This completes the computation of the ordering terms that arise from purely bosonic terms.

It remains to consider the fermionic terms. The possible ordering terms that may be generated are of the form $X\theta^2$ which arise from reordering $X^2P\theta^2$ and $X\theta^4$ terms. Let us first consider the terms with four fermion operators. The Clifford algebra identity (B.4) is required to simplify the second and third terms in eq. (78) of [8], to express $[M^{-a}, M^{-b}]$ in terms of the constraints. We will first argue that no ordering terms appear in the application of these identities. Notice that no term of order $X$ can be generated. The only possible terms would be of order $X\theta^2$. To see that these vanish notice that the relevant terms in the unordered expansion of $[M^{-a}, M^{-b}]$ appear schematically as

$$\theta_B\Gamma_1\theta_C\theta_D\Gamma_2\theta_E + \theta_D\Gamma_2\theta_E\theta_B\Gamma_1\theta_C\,, \tag{3.8}$$

where we have dropped the common prefactor, and $\Gamma_1$, $\Gamma_2$ are some products of the gamma matrices. This may be rearranged to

$$\theta_B\Gamma_1\theta_C\theta_D\Gamma_2\theta_E + \theta_E\Gamma_2\theta_D\theta_C\Gamma_1\theta_B\,, \tag{3.9}$$

without generating any ordering terms. The ordering terms proportional to $X\theta^2$ will then be proportional to the expression (3.9) with $\theta_C$ and $\theta_D$ contracted

$$\theta_B\Gamma_1\Gamma_2\theta_E + \theta_E\Gamma_2\Gamma_1\theta_B\,. \tag{3.10}$$

Then $\theta_E$ and $\theta_B$ can be reordered in the second term to cancel the contribution from the first term. Thus we see no ordering terms are generated in the application of the spinor identities to the four fermion terms.

Now consider the terms that can appear in $A_1+A_2+A_3$. It suffices to check whether eq. (86) of [8] can pick up ordering terms. This equation is:

$$\begin{aligned}C^{(1)} = {}& \frac{i}{2}\{X^a_A(\theta^A\gamma^b\theta_D) - X^b_A(\theta^A\gamma^a\theta_D)\}\varphi^D - \\ & -\frac{i}{4}\{X^a_A(\theta_B\gamma^b\theta_E) - X^b_A(\theta_B\gamma^a\theta_E)\}d^{ABC}c_{DC}{}^{E}\varphi^D + \\ & +\frac{i}{2}(\theta_D\gamma^{abd}\theta^D)X_{dE}\varphi^E - \frac{i}{4}(\theta_D\gamma^{abd}\theta_E)X_{dC}d_A^{\;DE}c_B^{\;AC}\varphi^B,\end{aligned} \tag{3.11}$$

which is to be understood as an unordered expression. The ordering terms will be proportional to terms from the contraction of $P$ with the $X$'s and from contractions of pairs of $\theta$'s. The terms from the first line of (3.11) cancel straightforwardly. Using (A.1$e$) one likewise finds cancellation of terms on the second line. For the first term on the third line of (3.11) the contractions vanish trivially. For the second term on the third line we use (A.3$a$) and (A.1$b$) to show cancellation of the terms. This completes the proof that no ordering terms appear in $A_1 + A_2 + A_3$.



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Finally we must consider $A_4$. Let us just consider the second term in (3.2) and factor out $\theta_0\gamma^b$. The only terms in this expression that can give us trouble are the $X^2P\theta$ terms and the $X\theta^3$ terms which could give ordering terms of the form $X\theta$. The computation reduces to showing the ordering terms between equations (93) and (94) of [8] cancel. Equation (94) gives the $X\theta^3$ terms

$$\begin{aligned}(D_4)_\alpha = & \frac{i}{2}(\gamma^a\theta_A)_\alpha(\theta_B\gamma\cdot X_C\theta_D)d^{AB}{}_E f^{CDE} + \\ & +\frac{i}{4}(\gamma^{ad}\theta_B)_\alpha X_{dA}(\theta_C\theta_D)c^{ECD}f^{AB}{}_E + \\ & +\frac{i}{4}\left[-(\gamma_d\theta_A)_\alpha(\theta_B\gamma^{ade}\theta_C)+(\gamma^{ed}\theta_A)_\alpha(\theta_B\gamma^a{}_d\theta_C)\right]X_{eD}d^{BC}{}_E f^{DAE} + \\ & +\frac{i}{2}(\gamma_d\theta_A)_\alpha(\theta_B\gamma^d\theta_C)X^a_D d^{BD}{}_E f^{ACE}\,. \end{aligned} \tag{3.12}$$

To simplify this equation, the Clifford algebra identity (B.4) must be applied to the third term in (3.12). To use this we must show the ordering terms in this third term vanish. This follows straightforwardly using the fact the $d_{ABC}$ is symmetric. To further simplify (3.12) as in [8] we must apply some group theory identities to the remaining terms. Using the fact that Tr $\gamma^a = 0$ and $c^A_{AB} = 0$, it can be shown that no ordering terms are generated in this process. We then can reduce $D_4$ to

$$\begin{aligned}(D_4)_\alpha = & -\frac{i}{4}(\gamma^{ad}\theta_C)_\alpha X_{dD}(\theta_B\theta_A)c_E{}^{CD}f^{EBA} - \\ & -\frac{i}{3}\left[(\theta_B\theta_A)\theta_{\alpha C}+(\theta_C\theta_A)\theta_{\alpha B}\right]X^a_D d^{BD}{}_E f^{CAE} - \\ & -\frac{i}{2}\left[(\gamma_d\theta_{[A})^\alpha(\theta_B\gamma^d\theta_{C]})\right]X^a_D d^{BD}{}_E f^{CAE}\,. \end{aligned} \tag{3.13}$$

The spinor identity B.5 must be applied to the third term in (3.13). Using the symmetries no ordering terms are generated in this process. Combining the resulting expression with the $X^2P\theta$ terms we then obtain

$$[\tilde{Q}^+_\alpha, \tilde{M}^{-a}] = i\theta_{\alpha C}X^a_D d^{CD}{}_E\varphi^E + \frac{i}{2}(\gamma^{ad}\theta_C)_\alpha X_{dD}c_E{}^{CD}\varphi^E\,. \tag{3.14}$$

Finally we must check no ordering terms appear when this expression is (a)symmetrically ordered. The terms generated by the first term vanish using the fact that $d_{ABC}$ is symmetric. The terms generated by the second term cancel when we apply the identities (A.2) and (A.1$b$). This completes the proof that quantum ordering terms do not appear at this order in $[M^{-a}, M^{-b}]$.

### 3.2. $[M^{-a}, H]$

The two possible nontrivial ordering contributions to this commutator come from $[\tilde{M}^{-a}, \mathcal{M}^2]$ and $[\tilde{Q}^+, \mathcal{M}^2]$. Let us ignore the fermionic contributions for the moment and consider $[\tilde{M}^{-a}, \mathcal{M}^2]$. It is straightforward to see the only nontrivial can appear at order $X^3$. There are three ordering contributions:

$$
\begin{aligned}
&[\frac{1}{2}d^{ABC}X^a_A P_B\cdot P_C, \frac{1}{2}f_{DEF}f^{D}_{\ GH}X^E\cdot X^G X^F\cdot X^H]\to \\
&\qquad\qquad\qquad\qquad 2d^{ABC}f_B^{\ DE}f_{CDG}(D-1)X^a_A X_E\cdot X^G\,,\\
&[\frac{1}{4}d^{ABC}f_B^{\ DE}f_C^{\ FG}X^a_A X_D\cdot X_F X_E\cdot X_G, P^2_H]\to \\
&\qquad\qquad\qquad\qquad -2d^{ABC}f_B^{\ DE}f_{CDG}(D-1)X^a_A X_E\cdot X^G\,,\\
&[\frac{1}{2}c^{ABC}P^a_A P_B\cdot X_C, \frac{1}{2}f_{DEF}f^{D}_{\ GH}X^E\cdot X^G X^F\cdot X^H]\to \\
&\quad 2(c^{ABC}f_{DAF}f^{D}_{\ BH}+c^{AB}_{\quad F}f_{DAB}f^{D}_{\ CH}+c^{AB}_{\quad H}f_{DAF}f^{D}_{\ CB})X^a_C X^F\cdot X^H\,.
\end{aligned}
\tag{3.15}
$$

The first two lines cancel against each other. Using the identities $c^{ABF}f_{DAB}\sim\eta_{DF}$ and A.1$c$, one finds the third line vanishes. This completes the proof that the ordering terms of bosonic origin vanish.

Now let us consider terms generated by the fermionic terms. Terms of order $\theta^2$ cancel by symmetry considerations. $c$-number terms cancel by $SO(9)$ invariance. The only non-trivial terms are again of order $X^3$ would arise from reordering $X^3\theta^2$ terms. This corresponds to properly ordering the terms in eq. (3.35d) of [7]. The possible non-zero ordering contributions are proportional to

$$
\begin{aligned}
&d^{ABC}f_A^{\ FG}f_C^{\ DE}X_{Fb}X_{Gc}X_{Dd}\eta_{BE}\mathrm{Tr}\,(\gamma^{abc}\gamma^d)\,,\\
&\qquad\qquad 2d^{ABC}f_C^{\ DE}f_E^{\ FG}X^a_A X^b_D X^c_F\eta^{bc}\eta_{BG}\,.
\end{aligned}
\tag{3.16}
$$

The first term vanishes since the $\mathrm{Tr}\,(\gamma^{abc}\gamma^d)=0$. The second term vanishes by symmetry. The massaging of the properly ordered terms in the commutator $[M^{-a}, H]$ using the identities of Appendix A, then proceeds precisely as in [7]. The final step is to show the unordered expression (3.39) of [7] does not give rise to ordering terms. This is straightforward to verify, using the Jacobi identity A.1$a$.

To complete the calculation we consider the commutator $[\tilde{Q}^+, \mathcal{M}^2]$. The only problematic terms will come from ordering terms of order $\theta^3$, which take the form

$$
f_{ABC}\gamma_a\theta^A(\theta^B\gamma^a\theta^C)\,.
\tag{3.17}
$$

It is clear the ordering terms which arise by contractions of pairs of $\theta$'s will vanish by symmetry. This completes the proof that the ordering terms in $[M^{-a}, H]$ vanish at this order, as required by Lorentz invariance.

## 4. Relation to Matrix theory

The key difference between the model of dWHN [1] and Matrix theory [2] is the treatment of the longitudinal momentum. In dWHN this is identified with the additional zero mode $P_0^+$ which satisfies a nontrivial commutation relation with $q^-$. This mode is crucial for constructing the boost generators of the previous sections. In Matrix theory, the size of the matrices $N$ is identified with the longitudinal momentum $P^+ = N/R$.[1] and the challenge is to construct the conjugate to this operator which would play the role of $q^-$ and allow us to construct boost generators

For finite values of $N$ it seems difficult to construct such an operator, as it maps between Matrix models with different values of $N$. However progress can be made in the large $N$ limit. At large $N$ we can consider a generalization of the ideas of Brezin and Zinn-Justin [12], and construct a renormalization group equation for the generating function of correlation functions $\mathcal{F}$. (See [13] for further work on this approach and more recent references.) This approach has been considered in the context of Matrix theory in curved space by Douglas [14]. The basic idea is to consider the renormalization group flow when one row and one column of the matrices is integrated out. In general, this should be equivalent to a deformation of the theory with $N$ fixed. This may be expressed as a renormalization group equation which takes the form

$$(N\frac{\partial}{\partial N} - \beta_i(\lambda_j)\frac{\partial}{\partial \lambda_i} + \gamma(\lambda_i))\mathcal{F} = r(\lambda_i)\,, \tag{4.1}$$

where the $\lambda_i$ are the coupling constants of operators added to the action. For the case at hand it is difficult to carry out this integration exactly, but the hope is that a systematic saddle point approximation can be developed.

However, if Matrix theory recovers eleven-dimensional Lorentz invariance in the large $N$ limit (with $N/R$ fixed), this rescaling of $N$ corresponds to a longitudinal boost. For Lorentz invariant physical quantities like S-matrix elements, the renormalization group equation should simplify to

$$N\frac{\partial}{\partial N}\mathcal{F} = 0\,, \tag{4.2}$$

where it is assumed the parameters labeling the longitudinal momenta are written as $P_i^+ = \alpha_i N/R$, with $\alpha_i$ held fixed in the derivative. For more general physical quantities $\mathcal{F}$, the fact that $N$ should appear only in the combination $N/R$ implies the renormalization group equation takes the form

$$(N\frac{\partial}{\partial N} + R\frac{\partial}{\partial R})\mathcal{F} = 0\,. \tag{4.3}$$

This equation can then be thought of as a definition for the conjugate of $N$, in terms of a derivative with respect to $R$.

The results for the Lorentz algebra of the previous section may then be carried over to Matrix theory by replacing $P_0^+$ with $N/R$ and $q^-$ with

$$q^- \to -i\frac{R^2}{N}\frac{\partial}{\partial R}\,. \tag{4.4}$$

[1]Here we have in mind the interpretation of Matrix theory as the description of M-theory on a compact light-like circle of radius $R$ [11, 3].

Related ideas in the context of a membrane in four dimensions have been discussed in [15]. The real challenge remaining is to verify (4.3) directly by performing an integration over a row and a column in the large $N$ limit.

The definition for $X_A^-$ (2.19) gives us the non-abelian version of the longitudinal coordinate as an operator written in terms of the transverse degrees of freedom and $R$. Assuming the above proposal is correct, it now becomes possible to construct states in Matrix theory localized in the longitudinal direction.

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## A. Area preserving diffeomorphism identities

The tensors $f_{ABC}$, $d_{ABC}$ and $c_{ABC}$ satisfy a number of identities in the large $N$ limit, up to $1/N^2$ corrections. These are derived using the completeness relations (2.12), symmetry, and integration by parts [7]

$$
\begin{aligned}
f_{[AB}{}^{E} f_{C]DE} &= 0\,, && \text{(A.1}a\text{)}\\
c_{ABC} + c_{ACB} &= 2\int d^2\sigma\sqrt{w(\sigma)}\frac{1}{\omega_A}\Delta Y_A(\sigma)Y_B(\sigma)Y_C(\sigma) = -2d_{ABC}\,, && \text{(A.1}b\text{)}\\
c_{DE}{}^{[A} f^{BC]E} &= 2\int d^2\sigma\sqrt{w(\sigma)}\frac{w^{rs}(\sigma)}{\omega_D}\partial_r Y_D(\sigma)\partial_s Y^{[A}(\sigma)\{Y^C(\sigma), Y^{B](\sigma)}\}=0, && \text{(A.1}c\text{)}\\
d_{ABC} f^{A}{}_{[DE} f^{B}{}_{F]G} &= \int d^2\sigma\sqrt{w(\sigma)}Y_C(\sigma)\{Y_{[D}(\sigma), Y_E(\sigma)\}\{Y_{F]}(\sigma), Y_G(\sigma)\} = 0\,, && \text{(A.1}d\text{)}\\
f_{A(B}{}^{E} d_{CD)E} &= \int d^2\sigma\sqrt{w(\sigma)}\frac{1}{3}\{Y_A(\sigma)\,,\, Y_B(\sigma)Y_C(\sigma)Y_D(\sigma)\} = 0\,, && \text{(A.1}e\text{)}\\
d_{EA[B} d_{C]D}{}^{E} &= \int d^2\sigma\sqrt{w(\sigma)}Y_A Y_{[B} Y_{C]} Y_D - \\
&\quad - \int d^2\sigma\sqrt{w(\sigma)}Y_A Y_{[B}\int d^2\rho\sqrt{w(\rho)}Y_{C]}Y_D = -\eta_{A[B}\eta_{C]D}. && \text{(A.1}f\text{)}
\end{aligned}
$$

For the special case of spherical topology, [7] found

$$
f_{AB}{}^{E} c_{ECD} = c_{EAB} f^{E}{}_{CD} - 2f_{BD}{}^{E} d_{ACE}\,. \qquad \text{(A.2)}
$$

Ezawa et al. [8] extended these identities to include

$$
\begin{aligned}
-c^{ABC} d_C{}^{EF} + 2c^{AC(E} d_C{}^{F)B} &= 4\eta^{A(E}\eta^{F)B}, && \text{(A.3}a\text{)}\\
\frac{1}{4}(c_C{}^{FE} c^{[AB]C} + c^{[A|F|}{}_C c^{B]CE}) &= -\frac{1}{2}\left(\frac{1}{\omega_A} - \frac{1}{\omega_B}\right)\frac{1}{\omega_C} f_C{}^{EF} f^{CAB} + && \text{(A.3}b\text{)}\\
&\quad + \frac{1}{2\omega_A\omega_B} f_C{}^{AB} f^{CEF}, && \text{(A.3}c\text{)}
\end{aligned}
$$

$$d_E{}^{AC}(c^{DEB} - 2d^{DEB}) - 2c^{D(C|E|}d_E{}^{A)B} = 4\eta^{AC}\eta^{BD}, \tag{A.3$d$}$$

$$d^{CG}{}_H d^{DH}{}_I f^{I[EF} f^{B]A}{}_G = -f^{D[EF} f^{B]AC}. \tag{A.3$e$}$$

## B. $SO(9)$ Clifford algebra identities

We review here some definitions and identities that are used above. The gamma matrices $\gamma^a{}_{\alpha\beta}$ ($a = 1, \ldots, 9$ ; $\alpha, \beta = 1, \ldots, 16$) are taken to be real and symmetric matrices. Using these gamma matrices we can construct an orthogonal complete basis for $16 \times 16$ real matrices

$$\left\{ I_{\alpha\beta} \ , \ \gamma^a{}_{\alpha\beta} \ , \ \gamma^{ab}{}_{\alpha\beta} \ , \ \gamma^{abc}{}_{\alpha\beta} \ , \ \gamma^{abcd}{}_{\alpha\beta} \ \right\}, \tag{B.1}$$

where we have defined

$$\gamma^{a_1 \cdots a_k} = \gamma^{[a_1}\gamma^{a_2} \cdots \gamma^{a_k]}. \tag{B.2}$$

$I$, $\gamma^a$ and $\gamma^{abcd}$ are symmetric, and $\gamma^{ab}$ and $\gamma^{abc}$ are antisymmetric with respect to the spinorial indices.

Some useful identities are:

$$\gamma^a \gamma^{b_1 \cdots b_k} = \gamma^{ab_1 \cdots b_k} + \sum_{l=1}^{k} (-)^{l-1} \delta^{ab_l} \gamma^{b_1 \cdots \check{b}_l \cdots b_k} \quad , \tag{B.3}$$

$$\begin{aligned} (\gamma^b)_{\alpha\beta}(\gamma_{ab})_{\gamma\delta} + (\gamma^b)_{\gamma\delta}(\gamma_{ab})_{\alpha\beta} + (\gamma^b)_{\alpha\delta}(\gamma_{ab})_{\gamma\beta} + (\gamma^b)_{\gamma\beta}(\gamma_{ab})_{\alpha\delta} - \\ -2I_{\delta\beta}(\gamma_a)_{\gamma\alpha} + 2I_{\alpha\gamma}(\gamma_a)_{\beta\delta} = 0\,. \end{aligned} \tag{B.4}$$

By multiplying (B.4) by $(\gamma^a)_{\delta\epsilon}(\theta^\beta_{[A}\theta^\gamma_B\theta^\epsilon_{C]})$ we obtain

$$(\gamma_d \theta_{[A})^\alpha (\theta_B \gamma^d \theta_{C]}) = \theta^\alpha_{[A}(\theta_B \theta_{C]}) \quad . \tag{B.5}$$

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